

Table -1

Eigenvalues (complex ;real positive,real negative ) of  
Non-Hermitian Hamiltonian for different values of  $w, L$  and  $W$  .

| n  | $w$  | $L$ | $W$ | Computed values           | Remarks     |
|----|------|-----|-----|---------------------------|-------------|
| 0  | 6    | 5.7 | 8   | 2.812 916                 | no relation |
| 1  |      |     |     | 14.464 644                | no relation |
| 2  |      |     |     | 24.190 191-55.906 544 i   | no relation |
| 10 |      |     |     | 47.296 364 -314.600 269 i | no relation |
| 0  | 13.7 | 5.7 | 8   | 4                         | $w = W + L$ |
| 1  |      |     |     | 12                        |             |
| 2  |      |     |     | 20                        |             |
| 10 |      |     |     | 84                        |             |
| 0  | 2.3  | 5.7 | 8   | 4                         | $w = W - L$ |
| 1  |      |     |     | 11.999 999 999            |             |
| 2  |      |     |     | 20                        |             |
| 10 |      |     |     | 83.999 999 999            |             |
| 0  | 13.7 | 8   | 5.7 | 2.85                      | $w = W + L$ |
| 1  |      |     |     | 8.559 999 999             |             |
| 2  |      |     |     | 14.249 999 999            |             |
| 10 |      |     |     | 59.849 999 999            |             |
| 0  | 2.3  | 8   | 5.7 | - 2.85                    | $w = L - W$ |
| 1  |      |     |     | - 8.55                    |             |
| 2  |      |     |     | - 14.25                   |             |
| 10 |      |     |     | -59.85                    |             |

# Complex Energy of Harmonic Oscillator under Non-Hermitian transformation of momentum with real wave function.

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For the first time in the literature of Quantum Physics, we present complex energy eigenvalues of non-Hermitian Harmonic Oscillator  $H = \frac{(p+iLx)^2}{2} + W^2 \frac{x^2}{2}$  with real wave function having positive frequency of vibration ( $w$ ) under some selective choice of  $L$  and  $W$ . Interestingly for the same values of  $L$  and  $W$ , if the frequency of vibration  $w$  in the real wave function is (some how) related as  $w = L \pm W$  or  $w = W - L$  then the same oscillator can reflect either pure positive or negative energy eigenvalues. The real energy levels are in conformity with the perturbative calculation.

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Key words- Positive frequency, real wave function, complex energy, real positive energy, real negative energy.

## I. Introduction

In quantum mechanics, the a well studied problem is Harmonic Oscillator characterized by the Hamiltonian is

$$H = \frac{p^2}{2} + W^2 \frac{x^2}{2} \quad (1)$$

The energy eigenvalue

$$E_n = (n + \frac{1}{2})W \quad (2)$$

and the corresponding wave function

$$\Psi_n(x) = [\frac{\sqrt{W}}{\sqrt{\pi n! 2^n}}]^{1/2} H_n(\sqrt{W}x) e^{-\frac{Wx^2}{2}} \quad (3)$$

are well known in the literature [1]. It should be borne in mind that the above positive definite system neither admits negative energy nor complex energy. In fact the concept of negative energy or complex energy in such Hamiltonian in real space is practically impossible. However if one can make a transformation  $x \rightarrow ix$  then the above Hamiltonian is transformed as

$$H = -[\frac{p^2}{2} + W^2 \frac{x^2}{2}] \quad (4)$$

which reflects negative energy i.e

$$E_n = -(n + \frac{1}{2})W \quad (5)$$

In this case wave function becomes

$$\Psi_n(x) = [\frac{\sqrt{W}}{\sqrt{\pi n! 2^n}}]^{1/2} H_n(\sqrt{W}ix) e^{\frac{Wx^2}{2}} \quad (6)$$

However negative spectrum on Harmonic Oscillator has germinated from the work of Bender, Hook and Klevansky [2] (hence forward BHK). The work of BVK[2] reported on a slightly modified form of above Hamiltonian as

$$H = p^2 + W^2 x^2 \quad (7)$$

saying that one can visualize the negative energy as

$$E_n = -(2n + 1)W \quad (8)$$

in two different ways namely (i) negative frequency  $W$  without changing its co-ordinate and (ii) making a co-ordinate transformation in complex plane as suggested above. Similar concept of negative frequency ( $W < 0$ ) is also reflected in the work of Fernandez[3] under the simultaneous transformation of momentum and co-ordinate[4]

. However Rath [4] suggested that in real space having positive frequency of vibration, it is still possible to realise negative energy under simultaneous transformation of co-ordinate and momentum. It is true that wave function having negative frequency converges in complex plane [3] but in real plane it diverges nicely. The nature of wave function having negative frequency for the oscillator in Eq(1) is

$$\Psi_n(x) = [\frac{i\sqrt{W}}{\sqrt{\pi n!} 2^n}]^{1/2} H_n(\sqrt{W}ix) e^{\frac{Wx^2}{2}} \quad (9)$$

One can see that wave function as in both the cases is not the same (see Eqs(9,6)). In fact the wave function in Eq(6) is more acceptable rather Eq(9). From the above discussions we notice that spectrum of Harmonic oscillator with or without transformation is real in nature, whether it is positive or negative. Now question arises can one see complex energy in Harmonic Oscillator?. Till now there is no literature on it. However the aim of this work is to show that under transformation of momentum ( $p \rightarrow p + iLx$ ) the Hamiltonian in Eq(1) can reflect complex energy levels in real space having real wave function with positive frequency of vibration.

## II. Hamiltonian in second quantization notation.

The Hamiltonian

$$H = \frac{(p + iLx)^2}{2} + W^2 \frac{x^2}{2} \quad (10)$$

considered here is  $PT$  symmetric in nature [5]. This type of Hamiltonian becoming a new model particularly after the work of Ahmed [6] particularly to reflect iso-spectral behaviour. Now to transform the Hamiltonian in second quantization form we express the co-ordinate and momentum as follows[1] :

$$x = \frac{(a + a^+)}{\sqrt{2w}} \quad (11a)$$

and

$$p = i\sqrt{\frac{w}{2}}(a^+ - a) \quad (11b)$$

where the creation operator,  $a^+$  and annihilation operator  $a$  satisfy the commutation relation

$$[a, a^+] = 1 \quad (12)$$

and  $w$  is an unknown parameter. Further in number space we have

$$a|n\rangle_w = \sqrt{n}|n-1\rangle_w \quad (13a)$$

$$a^+|n\rangle_w = \sqrt{n+1}|n+1\rangle_w \quad (13b)$$

$$\langle n|a^+a|n\rangle_w = n \quad (13c)$$

where the state vector  $|n\rangle_w$  can be expressed as

$$|n\rangle_w = \left[\frac{\sqrt{w}}{\sqrt{\pi}n!2^n}\right]^{1/2} H_n(\sqrt{w}x) e^{-\frac{wx^2}{2}} \quad (14)$$

Here we express  $H$  as

$$H = H_D + H_N \quad (15a)$$

where

$$H_D = \left[w + \frac{(W^2 - L^2)}{w}\right] \frac{(2a^+a + 1)}{4} \quad (15b)$$

and

$$H_N = U \frac{a^2}{4} + V \frac{(a^+)^2}{4} \quad (15c)$$

where

$$V = \left[-w + \frac{(W^2 - L^2)}{w} - 2L\right] \quad (15d)$$

$$U = \left[-w + \frac{(W^2 - L^2)}{w} + 2L\right] \quad (15e)$$

### III. Matrix Diagonalisation Method .

Here we use matrix diagonalisation method [6,7] to determine the eigenvalues of the above Hamiltonian. In eigenvalue relation we solve the

$$H\psi = E\psi \quad (16)$$

where

$$|\psi\rangle = \sum_m A_m |n\rangle_w \quad (17)$$

Here  $A_m$  's satisfy only three term recurrence relation as

$$R_m A_{m-2} + S_m A_m + T_m A_{m+2} = 0 \quad (18)$$

where

$$R_m = \frac{U}{4} \sqrt{m(m+1)} \quad (19a)$$

$$T_m = \frac{V}{4} \sqrt{(m+1)(m+2)} \quad (19b)$$

$$S_m = \frac{[w + (W^2 - L^2)](2n+1) - E}{4w} \quad (19c)$$

#### IV. Selective frequency $w = L \pm W$ or $w = W - L$ and Energy Levels .

In the above MDM calculation ,we consider arbitrary values of  $W$  and  $L$  to study the nature of eigenvalues of non-Hermitian Oscillator.

##### (i) Parameters added i.e $w = L + W$ .

Now using Harmonic Oscillator basis  $|n\rangle_w$  one can find from Eq(15b) that

$$E_n = (n + \frac{1}{2})W \quad (20a)$$

This result is in conformity with the perturbative calculation [4] and calculation involving similarity transformation [3].

##### (ii) Parameters subtracted i.e $w = L - W$ .

Similarly using Harmonic Oscillator basis  $|n\rangle_w$  one can find from Eq(15b) that

$$E_n = -(n + \frac{1}{2})W \quad (20b)$$

This result is in conformity with the perturbative calculation [4] .

##### (iii) Parameters subtracted i.e $w = W - L$ .

Similarly using Harmonic Oscillator basis  $|n\rangle_w$  one can find from Eq(15b) the same relation as in Eq(20b) i.e

$$E_n = (n + \frac{1}{2})W$$

This result is in conformity with the perturbative calculation .

## V. Result and Discussion.

In Table-I we present energy eigenvalues of non-Hermitian oscillator for different values of  $L, W$  and  $w$  .It is observed that for arbitrary values of  $w$  ,the eigenvalues are in general combination of real and complex. However if the same values of  $L$  and  $W$  are incorporated in  $w$  as  $w = L \pm W$  then one can obtain either pure positive eigenvalues (for  $+$ ;  $-$  sign) or negative eigenvalue ( for  $-$ ).In conclusion we present complex nature of energy eigenvalues of non-Hermitian Oscillator.In the opinion of author in designing materials /experiments in volving non-Hermitian oscillator , this paper will definitely be an asset to both experimentalists as well as theoretical physicists.

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